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Local Solvability of the Operator $u_{tt} + ia(t)u_x + b(t)u_t + c(t)u$

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1. The recent paper [4] of Nirenberg and Treves gives a characterization of local solvability for partial differential operators with simple characteristics. One can ask whether a similar characterization is possible for operators with multiple characteristics. We will consider a special case of this problem by deriving a condition which is equivalent to local solvability for the class of operators $Lu = u_{tt} + ia(t)\, u_x + b(t)\, u_t + c(t)u$.

In addition to presenting the results, our purpose is to demonstrate two closely related techniques which we think will be useful in studying solvability for more general operators. The first is the factorization of the total symbol of an operator into a product of first order pseudodifferential operators. The second is the construction of formal exponential null solutions involving fractional powers of a parameter. The second technique has already been developed in considerable detail by Flaschka and Strang in [2] where it is applied to the study of correctness of the Cauchy problem for operators with multiple characteristics.

The remaining sections are organized as follows. In Section 2, we consider the problem of writing L as the product of first order pseudodifferential operators with symbols of the form $\tau - \sum_{i\geqslant -1} \alpha_i(t)\, \xi^{-i/2}$. We find that this is possible near the origin when $a(0) \neq 0$. This construction is an extension of the *Puiseux series* to the noncommutative case of differential operators with variable coefficients. In Section 3, we state a condition on these factors and show it is sufficient for local solvability of L at the origin. Section 4 shows the necessity of this condition. Thus, when $a(0) \neq 0$ we obtain a characterization of local solvability of L. When $a(0) = 0$, this method is not available; therefore, in Section 5 we abandon it and prove directly that L is always locally solvable in this case. However, it would be preferable to give a unified treatment of both cases. In Section 6, we discuss one approach to this goal. We show how to construct some formal factorizations of L even when $a(0) = 0$; the factors no longer have the form assumed in Section 2. In special cases the factorization is rigorously valid and the condition of Section 3 correctly predicts that these operators are locally solvable. In general, however, our technique is purely formal.

2. We are given an operator $Lu = u_{tt} + ia(t)\, u_x + b(t)\, u_t + c(t)u$ with analytic coefficients. We would like to factor L into a product $L = L_1 L_2$ of first order pseudodifferential operators and thus replace the study of solvability of L by the study of solvability of the first order L_i. This step is analogous to the reduction to first order factors employed by Nirenberg and Treves in [4].

We will assume throughout this paper that $a(t)$ does not vanish identically: if it does, L is then obviously locally solvable. By taking the Fourier transform of L in the x variable, we can reduce our problem to factoring the ordinary differential operator $\hat{L}v = v'' - \xi a(t)v + b(t)\, v' + c(t)v$. Suppose first that the coefficients are constant. Then the problem is simply factoring the polynomial $p(\tau, \xi) = \tau^2 - a\xi + b\tau + c$ where $\tau = d/dt$. This problem reduces to finding the roots of p. We remark that in this factorization, and in all that follow, we will be interested only in constructing the factors for large $|\xi|$ since this is all that is needed to decide local solvability. The solution to the constant coefficient problem is obtained by constructing the Puiseux expansion of the roots of p. It will be useful to recall this construction. We make the initial guess that the roots are $\tau_{\pm}^{-1} = \pm a^{1/2}\xi^{1/2}$. Then $p(\tau_{\pm}^{-1}, \xi) = \pm a^{1/2} b \xi^{1/2} + c = O(\xi^{1/2})$. We can improve the accuracy of our guess by setting $\tau_{\pm}{}^0 = \pm a^{1/2}\xi^{1/2} + f_{\pm}$. Then

$$p(\tau_{\pm}{}^0, \xi) = \pm 2a^{1/2}\xi^{1/2} f_{\pm} + f_{\pm}{}^2 \pm a^{1/2}\xi^{1/2} b + c$$

so that the choice $f_{\pm} = \mp b/2$ results in $p(\tau_{\pm}{}^0, \xi) = O(1)$. Continuing by induction we find that there are $\alpha_{\pm}{}^i$ such that for $\tau_{\pm}{}^N = \sum_{-1 \leqslant i \leqslant N} \alpha_{\pm}{}^i \xi^{-i/2}$,

$$p(\tau_{\pm}{}^N, \xi) = O(\xi^{-N/2}), \tag{1}$$

$$p(\tau, \xi) - (\tau - \tau_+{}^N)(\tau - \tau_-{}^N) = O(\xi^{-N/2}). \tag{2}$$

It can be shown that the infinite sum $\tau_{\pm}{}^\infty$ converges for large $|\xi|$ and that

$$p(\tau_{\pm}{}^\infty, \xi) = 0, \tag{1'}$$

$$p(\tau, \xi) - (\tau - \tau_+{}^\infty)(\tau - \tau_-{}^\infty) = 0; \tag{2'}$$

however, the less sharp results (1) and (2) will suffice for our purposes.

We need to generalize this construction to the case of variable coefficients. Motivated by the Puiseux series, we set $p(\tau, \xi) = \tau^2 - \xi a(t) + b(t)\tau + c(t) = (\tau - \tau_1)(\tau - \tau_2)$ and make the initial guess $\tau_i^{-1} = (-1)^i\, a^{1/2}\xi^{1/2}$. Then $p - (\tau - \tau_1^{-1})(\tau - \tau_2^{-1}) = b\tau + c - (a^{1/2})'\xi^{1/2}$. Next try $\tau_i{}^0 = \tau_i^{-1} + \alpha_i{}^0$. Then

$$\begin{aligned} p &- (\tau - \tau_1{}^0)(\tau - \tau_2{}^0) \\ &= b\tau + c - [-(\alpha_1{}^0 + \alpha_2{}^0)\tau + (\alpha_2{}^0 - \alpha_1{}^0)\, a^{1/2}\xi^{1/2} + \alpha_1{}^0\alpha_2{}^0 - (a^{1/2})'\xi^{1/2} \\ &\quad + (\alpha_2{}^0)']. \end{aligned}$$

Suppose that $a(0) \neq 0$. Then near zero, we can find α_i^0 satisfying

$$\alpha_1^0 + \alpha_2^0 = -b,$$
$$a^{1/2}(\alpha_1^0 - \alpha_2^0) = -(a^{1/2})',$$

so that $p - (\tau - \tau_1^0)(\tau - \tau_2^0) = O(1)$. Continuing by induction we find that there are smooth functions $\alpha_i^j(t)$ such that for

$$\tau_i^N = \sum_{-1 \leqslant j \leqslant N} \alpha_i^j(t)\, \xi^{-j/2} \tag{3}$$

we have

$$p(\tau, \xi) - (\tau - \tau_1^N)(\tau - \tau_2^N) = O(\xi^{-N/2}). \tag{4}$$

To construct α_i^j knowing α_i^k for $k < j$, we need only solve a system of the form

$$\begin{aligned} a_1^j - \alpha_2^j &= A, \\ a^{1/2}(\alpha_1^j - \alpha_2^j) &= B, \end{aligned} \tag{5}$$

where A and B are functions of the known α_i^k and their derivatives. The hypothesis that $a(0) \neq 0$ makes this possible; on the other hand, note that if $a(0) = 0$, then Eq. (5) need not have a smooth solution.

We have proven the following result.

THEOREM 1. *Suppose that $a(0) \neq 0$. Then there is a neighborhood of the origin in which there exist τ_i^N of the form* (3) *such that*

$$L = \left(\frac{\partial}{\partial t} - \tau_1^N\left(t, \frac{\partial}{\partial x}\right)\right)\left(\frac{\partial}{\partial t} - \tau_2^N\left(t, \frac{\partial}{\partial x}\right)\right) + R$$

where R can be made arbitrarily smoothing in x by choosing a large enough N.

3. Suppose that we wish to solve $Lu = f$, where f is a function in $C_0^\infty(R^2)$. After a Fourier transform in x, this becomes

$$U'' - \xi a(t)U + b(t)\, U' + c(t)U = \hat{f}(\xi, t) \tag{6}$$

where $U(t, \xi) = \hat{u}(\xi, t)$. Although a second order equation cannot be solved exactly, we can obtain asymptotic solutions of (6) valid for large $|\xi|$. This is enough information to decide whether the Fourier transform can be inverted to give a solution u of the original problem.

It would be possible to use the WKB method directly in (6), but for equations of order higher than two, this becomes complicated. It is much easier in general to factor the left-hand side into first order operators. This is possible in the present case when $a(0) \neq 0$; according to Section 2,

$$\frac{d^2}{dt^2} - a(t)\xi + b(t)\frac{d}{dt} + c(t) = \left(\frac{d}{dt} - \tau_1\right)\left(\frac{d}{dt} - \tau_2\right) + O(1),$$

$$\tau_i = \pm a^{1/2}\xi^{1/2} + O(1) \tag{7}$$

in some neighborhood of zero. We will utilize (7) to construct a parametrix for L.

Initially, we consider only large values of $|\xi|$, $|\xi| \geqslant A > 0$ say. The ordinary differential operators $(d/dt) - \tau_i$ have the Green's functions

$$G_+^i(t, s; \xi) = \begin{cases} e^{\int_s^t \tau_i(\sigma,\xi)\,d\sigma}, & t \geqslant s \\ 0, & t < s \end{cases}$$

$$G_-^i(t, s; \xi) = \begin{cases} 0, & t \geqslant s \\ -e^{\int_s^t \tau_i(\sigma,\xi)\,d\sigma}, & t < s. \end{cases}$$

Define $F_+^i(t, \xi) = \operatorname{Re} \tau_i(t, \xi)$, $F_-^i(t, \xi) = \operatorname{Re} \tau_i(t, -\xi)$. Suppose now that the τ_i satisfy the following condition:

(M) There is an $\epsilon > 0$ such that either $\sup_{|t|<\epsilon, \xi>A} F_+^i < \infty$ or $\inf_{|t|<\epsilon,\xi>A} F_+^i > -\infty$ and either $\sup F_-^i < \infty$ or $\inf F_-^i > -\infty$.

Then when $\xi > A$ each factor $(d/dt) - \tau_i$ has a Green's function G^i such that

$$\sup_{|t|,|s|<\epsilon;\xi>A} |G^i(t, s; \xi)| < \infty. \tag{8}$$

Similarly, when $\xi < -A$ we can find a G^i such that

$$\sup_{|t|,|s|<\epsilon;\xi<-A} |G^i(t, s; \xi)| < \infty. \tag{8$'$}$$

Denote by $G^i(t, s; \xi)$ a Green's function with properties (8) and (8′) whenever $|\xi| > A$. Now form the integral operator

$$E_1 f(x, t) = \int_{|\xi|>A} e^{i\xi x}\, d\xi \int_{-\epsilon}^{\epsilon} G^2(t, r; \xi)\, dr \int_{-\epsilon}^{\epsilon} G^1(r, s; \xi) \hat{f}(\xi, s)\, ds$$

where $f(x, t)$ is C^∞ with compact support in the square $S = (-\epsilon, \epsilon)\, x(-\epsilon, \epsilon)$. (8) and (8′) imply that $E_1 f$ is a C^∞ function. Next, solve the equations $\hat{L}u = \hat{f}(\xi, t)$ when $|\xi| \leqslant A$. It is not important how this is done, as long as the result is piecewise continuous in ξ: for example one could use a finite number of converging power series in ξ by means of the regular perturbation expansions $u(\xi, t) = u_0(t) + (\xi - \alpha) u_1(t) + \cdots$. We form another integral operator from such solutions,

$$E_2 f(x, t) = \int_{|\xi|<A} e^{i\xi x} \hat{L}^{-1} \hat{f}(\xi, t) \, d\xi.$$

Finally, let $E = E_1 + E_2$. Then

$$LEf(x, t) = f(x, t) + Rf(x, t) \tag{9}$$

where $Rf(x, t) = \int_{|\xi|>A} O(1) \, e^{i\xi x} \, d\xi \int G^2 \, dr \int G^1 \hat{f} \, ds$; the $O(1)$ term is the error in the factorization (7).

Equation (9) reduces solving $Lu = f$ to solving $u + Ru = f$. The proof that $I + R$ is invertible in some neighborhood of the origin is standard, so we omit it. We have obtained the following solvability result.

THEOREM 2. *Let* $Lu = u_{tt} + ia(t) u_x + b(t) u_t + c(t)u$ *with* $a(0) \neq 0$. *Suppose that* (M) *holds. Then L is locally solvable at the origin.*

We conclude with a remark on condition (M). Note that whether it holds is independent of the $O(1)$ terms in τ_i. Therefore to check (M) we may consider the part of the τ_i involving only strictly positive powers of ξ. In the case of the operator L, this part is simply $\pm a^{1/2} \xi^{1/2}$, so that we see that (M) *is equivalent to the requirement that* Re $a^{1/2}$ *and* Im $a^{1/2}$ *do not change sign at zero.* Another equivalent formulation is that Im a not change sign at zero. In more general factorizations, however, more than one positive power of ξ will appear and the solvability condition will not reduce to the sign changes of a single function. (M) is therefore a generalization of the sign change condition familiar from the simple characteristics case.

4. In this section, we prove that if (M) does not hold, L is nonsolvable. It would be possible to base a nonsolvability proof on the factorization of Section 2 by proving the nonsolvability of one of the pseudodifferential factors; for brevity, we will deal with the operator L directly.

THEOREM 3. *Let L be as in Theorem* 2 *with* $a(0) \neq 0$. *If* (M) *does not hold, L is not locally solvable at the origin.*

We will use a modification of Hormander's method for showing non-solvability (see [3, Chapt. 6]). This method consists in the construction of a formal solution

$$u = e^{\xi \phi} \{ v^0 + (v^1/\xi) + (v^2/\xi^2) + \cdots \} \tag{10}$$

of ${}^tLu = 0$ such that Re $\phi \leqslant 0$ in a neighborhood of the origin, and Re $\phi = 0$

only at the origin. The definition of a formal solution in this context is that if the series is stopped at the N-th term, the resulting finite expression u_N should satisfy ${}^tLu_N = O(\xi^{-N})\, e^{\xi\phi}$.

The geometric optics expansion in (10) is a useful asymptotic solution for any operator with simple characteristics; however, it sometimes breaks down in the presence of multiple characteristics. For example, the only formal solution of $Lu = 0$ of that form is the trivial solution $u \equiv 0$. In [5] Strang noted that the appropriate analog of (10) for multiple characteristics is an expansion in fractional powers

$$u = e^{\sum_{-n\leqslant i<0}\xi^{-i/m}\phi_i} \cdot \sum_{i\geqslant 0} \xi^{-i/m} v_i\,. \tag{11}$$

Flaschka and Strang have shown in [2] that a nontrivial formal solution of the form (11) can always be constructed arbitrarily near a given point for any operator with characteristics of constant multiplicity.

In our case, the correct special case of (11) is easy to find. The expansion

$$u = e^{i\xi x+\xi^{1/2}\phi}\{v^0 + (v^1/\xi^{1/2}) + (v^2/\xi^1) + \cdots\} \tag{12}$$

is a formal solution of $Lu = 0$ if

$$\begin{gathered} \phi = \int_0^t (a(\tau))^{1/2}\, d\tau - x^2, \\ 2\phi_t v_t^0 + L\phi \cdot v^0 = 0, \\ 2\phi_t v_t^i + L\phi \cdot v^i + Lv^{i-1} = 0, \qquad i \geqslant 1. \end{gathered} \tag{13}$$

Since $a(0) \neq 0$, ϕ is a smooth function near the origin, and there are smooth solutions of the equations for the v^i. We observed in the last section that (M) is equivalent to $\operatorname{Re} a^{1/2}$ and $\operatorname{Im} a^{1/2}$ not changing sign at zero. Suppose that $\operatorname{Re} a^{1/2}$ changes sign, for example. Then since $a(t)$ is analytic, we have $\operatorname{Re}(a(t))^{1/2} = a_{2n+1}t^{2n+1} + O(t^{2n+2})$. Since there is a choice of sign in $a^{1/2}$ we may assume that $a_{2n+1} < 0$. Then

$$\phi(t, x) = (a_{2n+1}/2n + 2)\, t^{2n+2} - x^2 + O(t^{2n+3}).$$

Using the solution (12) with this choice of ϕ and letting $\xi \to +\infty$ in Hormander's argument, we conclude that ${}^tL$ is nonsolvable at the origin. If instead, $\operatorname{Im} a^{1/2}$ changes sign, we would let $\xi \to -\infty$ and again conclude that ${}^tL$ is nonsolvable. Of course, (M) holds for L if and only if it holds for ${}^tL$. Thus, we have proven Theorem 3.

We noted in Section 1 that the factorization of Section 2 and the construction of formal solutions of the form (11) are closely related. This is easily seen for L when b and c are identically zero; in this case, $\tau^2 - \xi a(t) = (\tau - \tau_1)(\tau + \tau_1)$ where τ_1 satisfies the Riccati equation $\tau_1' - \tau_1^2 = -\xi a(t)$ associated with the operator $\hat{L}$. Therefore $Le^{\int \tau_1(\sigma,\xi)d\sigma} = 0$ and if τ_1 is chosen as in Section 2, $e^{\int \tau_1(\sigma,\xi)d\sigma}$ is a solution of $Lu = 0$ of the form (11).

5. When $a(0) = 0$, the previous methods cannot be used to decide either solvability or nonsolvability at the origin. For example, let $Lu = u_{tt} + i\alpha t u_x$. Factorization as in Section 2 gives the root

$$\tau_1 = (\alpha\xi t)^{1/2} + 1/4t - (5/32\alpha^{1/2})\, t^{-5/2}\xi^{-1/2} + O(\xi^{-3/2})$$

which is highly singular at zero. Similarly, the expansion of Section 4 is not possible: the equation for the v^i is $2(\alpha t)^{1/2}\, v_t^i + L\phi \cdot v^i + Lv^{i-1} = 0$ and the zero in the coefficient of v_t^i means that smooth solutions may not exist. Of course, both expansions are also ambiguous because the function $(\alpha t)^{1/2}$ is not well defined. For now, we will abandon the previous methods and investigate solvability when $a(0) = 0$ separately.

THEOREM 4. *Let* $Lu = u_{tt} + ia(t)\, u_x + b(t)\, u_t + c(t)u$. *If* $a(0) = 0$, L *is locally solvable at the origin.*

To begin, let $a(t) = \alpha t^n$, α a complex constant, and $b \equiv c \equiv 0$. Taking the Fourier transform of $Lu = f$ gives

$$U'' - \alpha\xi t^n U = \hat{f}(\xi, t), \tag{14}$$

just as in (6). Consider the equation

$$v'' - t^n v = 0. \tag{15}$$

The Green's function $G(t, s; \xi)$ for (14) can be expressed in terms of two independent solutions Ei and Fi of (15), namely,

$$G(t, s; \xi) = \frac{\text{const}}{\lambda} \begin{cases} Ei(\lambda t)\, Fi(\lambda s) & t \geq s \\ Ei(\lambda s)\, Fi(\lambda t) & t < s; \end{cases} \quad \lambda = (\alpha\xi)^{1/(n+2)},\ |\arg \lambda| \leq \pi/(n+2).$$

We wish to choose Ei and Fi so that the Fourier transform in (14) can be inverted. Let $Ei(z) = z^{1/2} K_{1/(n+2)}(2z^{(n+2)/2})$. The asymptotic expansion for K_ν, see Watson [6], shows that $Ei(z) \sim (\pi/2)^{1/2}\, e^{-2z^{(n+2)/2}}$ when $|\arg z| \leq \pi/(n+2)$. We need the asymptotic behavior of Ei when $|\arg z - \pi| \leq \pi/(n+2)$. Assume that n is even. Then $Ei(-z)$ is also a solution of (15). From the series expansion of K_ν, we know that Ei has a Taylor series of the form $\sum a_i z^{(n+2)i} + z \sum b_i z^{(n+2)i}$. Therefore, $Ei(-z)$ is independent of $Ei(z)$. There can be at most one independent recessive solution of a second order equation in any sector of the complex plane. Since $Ei(z)$ is recessive in $|\arg z| \leqslant \pi/(n+2)$ it follows that $Ei(-z)$ is dominant there, and its leading order asymptotic expansion is $(\pi/2)^{1/2}\, e^{+2z^{(n+2)/2}}$. Thus $Ei(z)$ is dominant when $|(\arg z) - \pi| \leqslant \pi/(n+2)$. Let $Fi(z) = Ei(-z)$. When $|\lambda t|, |\lambda s| > M$ for some large M, $|\lambda G(t, s; \xi)|$ is uniformly bounded. Since $Ei(\lambda r)$ and $Fi(\lambda r)$ are bounded when $|\lambda r| \leqslant M$ we conclude that $\sup_{|t|,|s| \leqslant A; \xi} |G(t, s; \xi)| < \infty$ for any finite A. Thus, the Fourier transform is invertible, and L is locally solvable. The argument when n is odd is similar, but the choice of Fi is taken to be $Ei(ze^{\pm 2\pi i(n+1)/(n+2)})$ depending on the sign of $\operatorname{Im} \alpha$.

It is easy to show that the fundamental solution $Ef = \int_{-\infty}^{\infty} e^{i\xi x}\, d\xi \cdot \int G(t, s; \xi) \hat{f}(\xi, s)\, ds$ for $Lu = u_{tt} + i\alpha t^n u_x$ is a parametrix for any operator $L'u = Lu + b(t)\, u_t + c(t)u$. In this sense, L dominates any u_t and u terms. We conclude that L' is solvable.

Now consider the general operator L with a zero in the coefficient $a(t)$. We write it as $Lu = u_{tt} + i\alpha t^n A(t)\, u_x + b(t)\, u_t + c(t)u$ where $A(0) = 1$. To solve $Lu = f$, we take the Fourier transform:

$$U'' - \alpha\xi t^n A(t)\, U + b(t)\, U' + c(t)U = \hat{f}(\xi, t). \tag{16}$$

To construct a Green's function for (16), we attempt to find complementary solutions of the form $u(t) = Ei(\lambda\phi(t))$. $u'' = \lambda\phi'' Ei'(\lambda\phi) + \lambda^2(\phi')^2\, Ei''(\lambda\phi) = (\phi''/\phi')\, u' + \lambda^{n+2}(\phi')^2 \phi^n u$, so if $\lambda^{n+2} = \alpha\xi$ and $(\phi')^2 \phi^n = t^n A(t)$, u satisfies

$$u'' - \alpha\xi t^n A(t) u - (\phi''/\phi')\, u' = 0. \tag{17}$$

Note that there is a solution for ϕ of the form $t + O(t^2)$, so ϕ''/ϕ' is analytic near zero. Equation (17) has the Green's function

$$G(t, s; \xi) = \frac{1}{\lambda\phi'(s)} \begin{cases} Ei(\lambda\phi(t))\, Fi(\lambda\phi(s)), & t \geqslant s \\ Ei(\lambda\phi(s))\, Fi(\lambda\phi(t)), & t < s \end{cases}$$

where we choose Fi as when $A(t) \equiv 1$. It is easy to check that there is an $\epsilon > 0$ such that $\sup_{|t|,|s| < \epsilon; \xi} |G(t, s; \xi)| < \infty$; therefore

$$Ef(x, t) = \int_{-\infty}^{\infty} e^{i\xi x}\, d\xi \int G(t, s; \xi) \hat{f}(\xi, s)\, ds$$

is a C^∞ function whenever $f \in C_0^\infty((-\epsilon, \epsilon)\, x(-\epsilon, \epsilon))$, and $L'Ef = f$ where $L'u = u_{tt} + i\alpha A(t)\, t^n u_x - (\phi''/\phi')\, u_t$. As before, E is a parametrix for any operator differing from L' only in u_t and u terms. In particular, E is a parametrix for L; we conclude that L is locally solvable.

In order to simplify the statements of the results, we have assumed that the coefficients in L are all analytic. For completeness, we note that certain results can be obtained by the same methods if the coefficients are merely C^∞. Thus, (M) is still sufficient for solvability as long as $a(0) \neq 0$. However, our proof of solvability when $a(0) = 0$ requires that $a(t)$ vanish only to finite order at zero. Similarly, just as in [4], the proof of necessity of (M) requires Im a to vanish at zero to finite order.

6. We have found that when $a(0) \neq 0$, local solvability of L is equivalent to a sign change condition similar to the solvability criterion for simple characteristics, whereas when $a(0) = 0$, L is always locally solvable. The hypothesis that $a(0) \neq 0$ enables us to carry out certain constructions, but it does not appear to be a fundamental property of L. Therefore, we think it is important to state these solvability results in a form that does not invoke this *ad hoc* hypothesis. This section is a preliminary investigation of this possibility.

At the beginning of Section 5, we attempted to factor the operator $Lu = u_{tt} + i\alpha t u_x$. The method of Section 2 led to the factors

$$\tau^\pm = \pm(\alpha t\xi)^{1/2} + 1/4t \mp 5/(32\alpha^{1/2})\, t^{-5/2}\xi^{-1/2} + O(\xi^{-3/2})$$

which become singular at zero. Furthermore, suppose we know that the expansion τ^+ is an asymptotic representation of some factor when $t > \epsilon > 0$; then we do not know what expansion $\tau^\pm$ corresponds to this factor when $t < -\epsilon$. Thus, if L can be factored, the symbols of the factors cannot have uniformly valid asymptotic expansions of the form $\tau - \sum_{j\geqslant -1} \alpha_j(t)\, \xi^{-j/2}$ near zero. Difficulties of this type occur in turning point problems and in singular perturbations; see for example Cole [1]. Suppose we now make no assumptions on the form of the factors of L. After a Fourier transform, we have the problem

$$\begin{aligned}\hat{L}U &= U'' - \alpha\xi t U = ((d/dt) - A(t,\xi))((d/dt) + A(t,\xi))U \\ &= U'' + (A_t - A^2)U.\end{aligned}$$

This representation will be valid if A satisfies

$$A_t - A^2 = \alpha\xi t.$$

This is the Riccati equation corresponding to $\hat{L}$; therefore it has the solutions $(\alpha\xi)^{1/3}[Ai'([\alpha\xi]^{1/3}t)/Ai([\alpha\xi]^{1/3}t)]$ where Ai is any solution of Airy's equation $u'' - tu = 0$. Let us choose for Ai the Airy integral $\int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} e^{zt - t^3/3}\, dt$. It is shown in [6] that this integral has the leading order asymptotic expansion

$$Ai(z) \sim (1/z^{1/4}(4\pi)^{1/2})\, e^{-(2/3)z^{3/2}}, \qquad |\arg z| < \pi \tag{18}$$

where the branch of $z^{1/2}$ is chosen with a branch cut on arg $z = \pi$ and so that $x^{1/2} > 0$ for $x > 0$. Thus, $A(t, \xi)$ has the asymptotic expansion

$$A(t, \xi) \sim (\alpha\xi t)^{1/2} + O(1) \tag{19}$$

when $\xi^{1/3}t$ is large. The branch of $z^{1/2}$ in (19) is the same as in (18).

Equation (19) solves the connection problem we noted earlier: it tells us how to match the Puiseux expansions of Section 2 which can be constructed on either side of zero. We know that $Ai(0) \neq 0$; therefore there exist α for which $A(t, \xi)$ is analytic. For such α, (19) shows that the factors $\tau \pm A(t, \xi)$ of L satisfy (M). The argument of Section 3 is applicable, and we can again conclude that L is locally solvable. However, for general values of α, poles appear in $A(t, \xi)$; these poles occur in the $O(1)$ terms in (19). For these α, the argument is incomplete. The validity of the factorization in this case requires further study.

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